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An Adaptive Stabilization Scheme for Autonomous System Oscillations[1]

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Abstract—A smooth autonomous system of general form is considered. A global family of nondegenerate periodic solutions by the parameter h is constructed; the period varies monotonically on this family. The problem of stabilizing the oscillations of the reduced controlled system is solved. A smooth autonomous control law with a parameter depending on h is applied, and an attracting cycle is constructed. The results are concretized for an nth-order differential equation. The relation of these results with the conclusions obtained for the reversible mechanical system is established. An adaptive control scheme for the reduced conservative system is proposed to stabilize any oscillation of the family. Some applications are presented.

Keywords: autonomous system, nondegenerate periodic solution, global family, Lyapunov center theorem, adaptive scheme, attracting cycle, natural stabilization

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1. INTRODUCTION

Consider the problem of controlling the oscillations (periodic solutions) of an autonomous system of general form. In the nondegenerate case, the following alternative is valid: either a cycle (isolated oscillation) or a family of oscillations. On a family of nondegenerate oscillations, the period monotonically depends on the family parameter h. Therefore, when stabilizing a family of oscillations, it is natural to look for a control law with a parameter depending on h. In an adaptive control system, the "controller automatically changes its structure or *its parameters depending on the changes of plant parameters* or disturbance properties" [1, p. 108].

In this paper, we apply an *autonomous* control law in which the controller parameter is found depending on the parameter h of the stabilized oscillation of the family: the controller has *adaptivity*. In this sense, the control scheme is said to be adaptive.

In the example

$$\ddot{x} + \omega^2 x = \varepsilon u, \quad u = (1 - Kx^2)\dot{x}, \tag{1}$$

the control law u contains the parameter K; ω is the frequency of a linear oscillator. For $\varepsilon = 0$, equation (1) admits the family of oscillations $x = A\cos\varphi$ with amplitude A and energy $h = \omega^2 A^2/2$. Let $K = 2\omega^2/h^*$ in (1) to stabilize the selected oscillation family with energy $h = h^*$. Then an attracting cycle close to the oscillation of the linear oscillator with energy h^* is implemented in (1). The formula for K is valid for any oscillation of the family. An adaptive stabilization scheme is applied.

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Equation (1) is called a van der Pol oscillator. The van der Pol equation [2] is obtained from (1) for $K = 1$: in a regerative receiver, ω is tuned to the frequency of the received signal.

A linear oscillator admits a family of isochronous oscillations. In a mathematical pendulum, the oscillation period is a monotonic function of the family parameter. Such a family of oscillations is called nondegenerate [3].

When studying oscillations in a multidimensional autonomous system, it is necessary to describe the entire set of nondegenerate oscillations. The problem of the global family of oscillations arises: existence, construction, and properties. Besides the interest for the theory of ordinary differential equations (ODEs), the knowledge of the global family ensures the complete solution of the control problem.

The main point about the global family is the possibility of continuing any local nondegenerate family of periodic solutions to the global family on which the family parameter will take all possible values for the solutions of the family. The global family for a particular case was constructed in [3]. An adaptive stabilization scheme for the oscillations of a reduced system on the plane was presented in [3]: an attracting cycle was constructed.

A global family is described by a reduced system whose dimension coincides with the dimension k of this family. In the general case (a system of n differential equations), we have $1 < k \leqslant n$. The existence of the global family, the isolation of the reduced system, and the design of an adaptive oscillation stabilization scheme for this system are considered below. The results are concretized for an nth-order differential equation. Also, we relate these conclusions to the results for systems possessing the properties of reversibility and conservativeness.

Some applications are presented. The three-body problem was introduced by L. Euler in 1764 when studying the motion of the Moon; it was published later in [4]. We demonstrate the construction of the global families of periodic orbits in this problem and apply the adaptive scheme for orbital stabilization. V.V. Beletskii's problem [5] describes the oscillations of a satellite in the elliptical orbit plane. In a particular case (a circular orbit), we stabilize any oscillation of the satellite using the adaptive control scheme.

2. GLOBAL FAMILY THEOREM

Consider a smooth autonomous equation of the form

$$\dot{z} = Z(z), \quad z \in \mathrm{R}^n. \tag{2}$$

Let $z(z_1^0, \ldots, z_n^0, t)$ denote the general solution of this equation, where $z^0 = (z_1^0, \ldots, z_n^0)$ is an initial point (for $t = 0$). A T-periodic solution of system (2) exists under the necessary and sufficient condition

$$f \equiv z(z_1^0, \ldots, z_n^0, T) - z^0 = 0. \tag{3}$$

Assume that equation (3) has a solution $z^0 = z^*$, $T = T^*$, not coinciding with the trivial equilibrium: $Z(z^*) \neq 0$. Then equation (3) admits a family of solutions in the parameter γ :

$$z^0 = z^*(\gamma), \quad T = T^*, \tag{4}$$

where γ is the shift of the initial time instant along the trajectory. In addition, the rank Ra of the functional (Jacobi) matrix A_f for the function f with the parameter T satisfies the inequality $\mathrm{Ra} \leqslant n - 1$ at the point z^* for $T = T^*$.

The case $\mathrm{Ra} = n - 1$ is said to be nondegenerate for a periodic solution; for details, see [3]. The case $\mathrm{Ra} = n - 1$ will be considered below.

In the nondegenerate case, the matrix A_f has a simple zero eigenvalue or a kth-order Jordan block of zero eigenvalues, $1 < k \leqslant n$. In the first situation, equation (3) admits a unique solution of the form (4). Therefore, the autonomous equation (2) admits an isolated periodic solution, i.e., a cycle with the period T^*.

The second situation is called the case of a family of periodic solutions.

In the neighborhood of the solution of equation (4), we introduce the linear system

$$\xi_s \equiv \frac{\partial f_s}{\partial z_1^0} dz_1^0 + \ldots + \frac{\partial f_s}{\partial z_n^0} dz_n^0 + \frac{\partial f_s}{\partial T} dT = 0, \quad s = 1, \ldots, n, \\ f = (f_1, \ldots, f_n), \tag{5}$$

which is valid for arbitrary γ. The matrix A_f in (5) has rank $\mathrm{Ra} = n - 1$. The cycle is obtained from the simple zero eigenvalue: in (5) we have the solution $dz_1(\gamma) = \ldots = dz_n(\gamma) = 0$, $dT = 0$.

The family case is implemented under $dT \neq 0$. The derivatives

$$\frac{df_s}{dT} = Z_s(z^*(\gamma)), \quad s = 1, \ldots, n,$$

are calculated at the point (4) and ensure that the rank of the augmented matrix in (5) is equal to the number $(n-1)$. The values $dz_1^0(\gamma), \ldots, dz_k^0(\gamma)$ are computed from (5) as linear functions of dT. Varying dT yields a family of solutions. The period T changes from solution to solution, i.e., it is a function of the scalar parameter $h : z^0 = z^0(\gamma, h)$, $T = T(h)$. Thus, the local h-family is found from system (5). The derivative is $T'(h^*) \neq 0$ for the parameter value $h = h^*$. Therefore, on the family, the period $T(h)$ changes monotonically together with the family parameter h. In this sense, the h-family is nondegenerate [3].

Definition 1. A family of periodic solutions of equation (2) is said to be nondegenerate if the period $T(h)$ on this family monotonically depends on the parameter h.

According to the presentation, the point $z^0 = z^*$ of a periodic solution of equation (2) has the property of leading to a nondegenerate family of periodic solutions. The reference periodic solution also belongs to the family. In this sense, it is said to be nondegenerate. Any periodic solution of a nondegenerate family is also nondegenerate.

Definition 2. A solution belonging to a nondegenerate family of periodic solutions called a nondegenerate solution.

A nondegenerate periodic solution can be continued in the period T or, equivalently, in the family parameter h. This is called *local extensibility property*. A nondegenerate periodic solution is continued simultaneously toward increasing and decreasing the period.

The concept of a global family of periodic solutions was introduced in [3].

Definition 3. A nondegenerate family of periodic solutions on which the parameter h takes all possible values for the family solutions is called a global family.

When reducing the matrix A_f to the canonical form, system (5) decomposes into two subsystems; one subsystem, with a zero kth-order Jordan block, leads to a family of periodic solutions, whereas the other subsystem has a zero solution in (5). Therefore, in the new variables, a family of periodic solutions is described by k variables. In the phase space, a global family is represented by a connected k-dimensional set of points.

In the case $\mathrm{Ra} = n - 1$ (nondegenerate for a periodic solution), we have the following result.

Theorem 1. *Assume that equation* (2) *admits a nondegenerate periodic solution. Then it extends in the period T to a global family Σ. On Σ the period $T(h)$ monotonically depends on the family* *parameter h. The family* Σ *fills the global domain* $\hat{\Sigma}$; Σ *is described by a reduced system of order k. For the points of the domain* $\hat{\Sigma}$, *the rank is* $\mathrm{Ra} = n-1$; *on its boundary* $\partial\hat{\Sigma}$, *the condition* $\mathrm{Ra} = n-1$ *fails.*

Proof. A nondegenerate periodic solution has local extensibility. This property is independent of the dimension k of the Jordan block with zero eigenvalues of the matrix A_f. Therefore, the proof of Theorem 1 coincides, within the dimension of the zero kth-order Jordan block and minor editorial changes, with that of [3, Theorem 1] for the case $k = 2$. The reduced system is described by a kth-order system.

Remark 1. When approaching the boundary $\partial\hat{\Sigma}$, the derivative $T'(h)$ may tend to zero, infinity, or cease to exist.

Remark 2. System (2) may simultaneously have several global families of periodic solutions with the same k. Also, there are no obstacles to the simultaneous existence of global families with different k.

3. ADAPTIVE CONTROL SCHEME FOR THE REDUCED SYSTEM

According to Theorem 1, the global family of periodic solutions $\Sigma = \{\varphi_s(h,t)\}$ can be described by a reduced system in $\mathrm{R^k}$:

$$\dot{x}_s = X_s(x_1, \dots, x_k), \quad s = 1, \dots, k. \tag{6}$$

The problem is to stabilize any $T(h)$-periodic oscillation φ from the family Σ chosen by the value of the parameter h. We apply a control law containing the parameter K, assigning the value of K depending on $h : K = K(h)$. The control law is defined by a smooth function F of the variables $x_1, \dots, x_k$ that acts with a small gain ε of the controller signal. In the controlled system

$$\dot{x}_s = X_s(x_1, \dots, x_k) + \varepsilon F_s, \quad s = 1, \dots, k, \tag{7}$$

and the stabilized oscillation will be ε-close to the oscillation of system (7). The parameter K in the control law ensures the existence of a periodic solution in (7) identically in h. Then the oscillation with the parameter $h = h^*$ is stabilized by substituting into the control function F the number $K = K(h^*)$ for which $dK(h^*)/dh \neq 0$. In the special case $k = 2$, the adaptive control scheme was implemented in [3].

The stabilization problem is posed in small. It is solved by constructing an attracting cycle. Therefore, we will find existence conditions for such a cycle and calculate its characteristic exponent (CE). These problems are solved in the neighborhood of the reference (basic) oscillation $x = \varphi(h^*, t)$ by a linearized system. The control value F_s is computed on the reference oscillation. The deviations from the reference oscillation and the number ε are considered of the same order.

Letting $\Delta_s = x_s - \varphi_s$, $s = 1, \dots, k$, in system (7), we write equations for the variables Δ_s. In the linear approximation with respect to Δ_s, the resulting equations contain the variations of δx_s. Then it is necessary to analyze the periodic solution of the controlled system

$$\delta\dot{x}_s = p_{s1}\delta x_1 + \ldots + p_{sk}\delta x_k + \varepsilon F_s, \quad s = 1, \dots, k. \tag{8}$$

System (8) coincides with the system derived from (7) for the increments Δ_s within the nonlinear terms. The cycle of system (8) is obtained from the linearized system (7) in the neighborhood of the reference oscillation.

On the manifold $\hat{\Sigma}$ the variables

$$\delta x_j = \frac{\partial x_j}{\partial h}, \quad j = 1, \dots, k,$$

are functions of h and t. We denote by $\{y_1, \ldots, y_k\}$ the solution of the adjoint system. Then the expression

$$y_1\delta x_1 + \ldots + y_k\delta x_k = \text{const} \tag{9}$$

is the first integral of the linear homogeneous system in (8). The set of these integrals are used to reduce the homogeneous linear system with periodic coefficients in (8) to a system with constant coefficients. In the case of the zero kth-order Jordan block, we obtain the equations

$$\dot{u}_1 = 0, \quad \dot{u}_2 = -u_1, \ldots, \dot{u}_k = -u_{k-1}. \tag{10}$$

All solutions of the adjoint system have the form

$$\begin{aligned} &y_{s1} = \psi_{s1}, \\ &y_{s2} = t\psi_{s1} + \psi_{s2}, \\ &\ldots \\ &y_{sk} = \frac{t^{k-1}\psi_{s1}}{k-1} + \ldots + t\psi_{s,k-1} + \psi_{sk}, \end{aligned} \tag{11}$$

where ψ_{sj} are functions with the period $T(h)$. Then the Lyapunov transform is given by

$$u_j = \psi_{1j}\delta x_1 + \ldots + \psi_{kj}\delta x_k, \qquad j = 1, \ldots, k, \tag{12}$$

with a nonsingular periodic matrix $||\psi_{sj}(h,t)||$. Hence, the corresponding derivatives are

$$\dot{u}_j = \psi_{1j}\frac{d\delta x_1}{dt} + \ldots + \psi_{kj}\frac{d\delta x_k}{dt} + \dot{\psi}_{1j}\delta x_1 + \ldots + \dot{\psi}_{kj}\delta x_k, \quad j = 1, \ldots, k.$$

As a result, the controlled system (8) in the variables u_j differs from (10) by the inhomogeneous terms:

$$\begin{aligned} &\dot{u}_1 = \varepsilon(\psi_{11}F_1 + \ldots + \psi_{k1}F_k) = \varepsilon\hat{F}, \\ &\dot{u}_j = -u_{j-1} + \varepsilon(\psi_{1j}F_1 + \ldots + \psi_{kj}F_k), \quad j = 2, \ldots, k. \end{aligned} \tag{13}$$

For $T = T^*$, system (13) defines a mapping $t : 0 \to T^*$ on the manifold $\hat{\Sigma}$. For $\varepsilon = 0$, the mapping admits a zero fixed point. Given $\varepsilon \neq 0$, a fixed point exists under the necessary conditions in the form of the amplitude equation

$$I(h) \equiv \int_0^{T^*} \hat{F}dt = \int_0^{T^*} \sum_{s=1}^{k} \psi_{s1}F_s dt = 0; \tag{14}$$

for details, see [6, Ch. VI, § 8, p. 413, §9, p. 417]). Equation (14) is with respect to the unknown h, with the function $\hat{F}$ containing the parameter $K = K(h^*)$ along with h. A particular form of (14) is provided in the example below for (21). The simple root $h = h^*$ of the amplitude equation (14) corresponds to an isolated periodic solution of system (13), hence, that of system (8). System (8) describes the cycle of system (7) within the nonlinear terms (in the neighborhood of the oscillation under study). Therefore, the cycle of system (7) is derived from the amplitude equation (14). The inequality $dI(h^*)/dh \neq 0$ is a sufficient condition for the existence of this cycle.

The CE of the cycle are found from the equations in variations. In the case of the zero kth-order Jordan block, a single number α is calculated to determine the CE [6, Ch. 3, § 11]. It shows the change in the variation over the period. The variations are found as the derivatives of the functions u_s with respect to the parameter h.

The first equation of system (13) has the form

$$\dot{u}_1(h^*,t) = \varepsilon\left[\hat{F}_* + \sum_{s=1}^{k}\left(\frac{\partial \hat{F}}{\partial x_s}\frac{\partial x_s}{\partial h}\right)_* \Delta h + \ldots\right],$$

where the asterisk denotes the values calculated for $h = h^*$. Then the increment Δh satisfies

$$\frac{d(\Delta u_1)}{dt} = \varepsilon\left[\sum_{s=1}^{k}\left(\frac{\partial \hat{F}}{\partial x_s}\frac{\partial x_s}{\partial h}\right)_* \Delta h + \ldots\right],$$

and we obtain the following equation for the derivative:

$$\frac{d}{dt}\left(\frac{\partial u_1}{\partial h}\right)_* = \varepsilon\sum_{s=1}^{k}\left(\frac{\partial \hat{F}}{\partial x_s}\frac{\partial x_s}{\partial h}\right)_* = \varepsilon\frac{\partial \hat{F}(h^*,t)}{\partial h}.$$

The change in the derivative over the period leads to the CE of the cycle

$$\left(\frac{\partial u_1}{\partial h}\right)_* = \frac{\varepsilon}{T^*}\int_0^{T^*}\frac{\partial\left(\sum\limits_{s=1}^{k}\psi_s(h^*,t)F_s(h^*,t)\right)}{\partial h}dt.$$

Thus, the CE α of the cycle is given by

$$\alpha = \frac{\varepsilon}{T^*}\int_0^{T^*}\frac{\partial \hat{F}(h^*,t)}{\partial h}dt. \tag{15}$$

Theorem 2. *For the reduced controlled system* (7)*, the cycle stabilization problem is solved for any chosen parameter value $h = h^*$ by a smooth control function F acting with a small gain of the controller signal. A sufficient condition for cycle stabilization is the inequality $dI(h^*)/dh < 0$ imposed on the root of the amplitude equation* (14)*. The characteristic exponent of the cycle is calculated using formula* (15)*.*

Theorem 2 is applied to all oscillations of the global family. The control law contains the parameter h, is found from the amplitude equation holding identically in h, and is designed by generalizing the universal control from [7]. In the case $k = 2$, several particular control laws satisfying Theorem 2 were given in [3]. For the nth-order equation, the adaptive control law is presented in Section 4.

Remark 3. According to Theorem 2, the cycle stabilization problem is solved by the control function $\hat{F}$. Therefore, we can choose in system (8), e.g., the control function $\hat{F}$ with $F_s \equiv 0; s = 2,\ldots,k$. For the second-order equation ($k = 2$), the control function is applied with $F_1 \equiv 0,\ F_2 \neq 0$.

4. nTH-ORDER DIFFERENTIAL EQUATION

An nth-order Jordan block with the zero eigenvalues of the matrix A_f is the only case for a single nth-order differential equation

$$x^{(n)} = X(x, x', \ldots, x^{(n-1)}), \tag{16}$$

where $x^{(j)}$ denotes the jth derivative of x.

Indeed, equation (16) turns into the system

$$\begin{aligned} \dot{x}_s &= x_{s+1}, \quad s = 1, \ldots, n-1, \\ \dot{x}_n &= X(x_1, \ldots, x_n), \end{aligned} \tag{17}$$

where the functions in (2) are $X_s = x_{s+1}$, $s = 1, \ldots, n-1$. Therefore, due to equation (3), the matrix A_f represents an nth-order Jordan block with zero eigenvalues.

Thus, if system (17) has a nondegenerate periodic solution, it belongs to the global family of $T(h)$-periodic solutions on which the period monotonically depends on the parameter h (Theorem 1). The solutions are defined in the space $(x, x', \ldots, x^{(n-1)})$, and the periodic solution of equation (16) is denoted by $x = \varphi(h, t)$.

Let us formulate the following problem: it is required to stabilize the solution of equation (16) selected by a value $h = h^*$. For this purpose, we consider system (7) with a vector control function $F = (F_1, \ldots, F_n)$ satisfying the amplitude equation (14) with a simple root. Theorem 2 is applied with a scalar control function $\hat{F}$ formed from the coordinates of the vector F. In the particular case of system (7) (i.e., equations (17)), we take $F_1 = \ldots = F_{n-1} = 0$ and the explicit-form function F_n. As a result, the controlled system is described by

$$x^{(n)} = X(x, x', \ldots, x^{(n-1)}) + \sigma\varepsilon[1 - Kx^2]x', \tag{18}$$

where the control value F_n acts with a small gain and the number σ is 1 or (-1). The coefficient K is assigned depending on the value of the parameter $h : K = K(h)$. To stabilize the oscillation with $h = h^*$, we let $K = K(h^*)$.

Thus, the adaptive control scheme is designed.

The control law applied in (18) is an analog of the universal control proposed in [7]. It satisfies the amplitude equation (14), which has a simple root for almost all points of the family in the parameter h.

The function $K(h)$ is obtained from the amplitude equation (14) holding identically in h. This function is calculated using the formula

$$K(h) = \frac{\int\limits_0^{T(h)} \psi_{nn}(h, t)\varphi'(h, t)dt}{\int\limits_0^{T(h)} \varphi^2(h, t)\psi_{nn}(h, t)\varphi'(h, t)dt};$$

the function $\psi_{nn}(h, t)$ is taken from (12). Then we have

$$\frac{dI(h^*)}{dh} = \chi\nu(h^*), \quad \chi = \frac{dK(h^*)}{dh}, \quad \nu(h^*) = \int\limits_0^{T(h^*)} \psi_{nn}(h^*, t)\varphi'(h^*, t)dt \tag{19}$$

for the function $I(h)$ (14) at the point $h = h^*$.

Theorem 3. *If the nth-order differential equation admits a nondegenerate periodic solution, it belongs to the global family in the scalar parameter h. The solution with the parameter value $h = h^*$, $\chi\nu \neq 0$, is stabilized by the adaptive control scheme* (18) *with $K = K(h^*)$: the sign of the number σ ensures the attraction to the cycle.*

Remark 4. By writing equation (16) as (17) and applying the adaptive control system of Section 4 to (17), we establish the attraction to the cycle in the space $(x, x', \ldots, x^{(n-1)})$ in equation (18).

5. SYMMETRIC PERIODIC MOTIONS

An autonomous system of the general form (2) may have additional properties such as conservativeness or reversibility, be written in the Hamiltonian form, etc. In this case, Theorem 1 remains valid as well. However, for some systems, this theorem needs to be clarified.

In what follows, the concept of a global family of periodic solutions of ODEs will be concretized for symmetric periodic motions.

A reversible dynamic system with the phase vector z and a nondegenerate mapping G has spatiotemporal symmetry in the sense of invariance with respect to the transformation $(z, t) \to (Gz, -t)$. It describes models in various fields of knowledge; see the survey in [8]. In the case

$$G = \left\| \begin{matrix} I_l & 0 \\ 0 & -I_n \end{matrix} \right\|, \quad l \geqslant n$$

(I_j is an identity matrix of dimensions $(j \times j)$), we obtain a reversible mechanical system [9]. The phase space of this system is described by vectors u and v such that $\dim u = l$, $\dim v = n$, and the symmetry transformation is $(u, v, t) \to (u, -v, -t)$. In mechanics, u and v are usually taken to be the vectors of generalized coordinates (quasi-coordinates) and generalized velocities (quasi-velocities), respectively. The set $M = \{u, v : v = 0\}$ is called the fixed set of a reversible mechanical system.

The phase portrait of a reversible mechanical system is symmetric with respect to the set M. The trajectories intersecting M are called symmetric. The twofold intersection of the set by a trajectory leads to a symmetric periodic motion (SPM). On an SPM of period $T/2$, the trajectory intersects the set M at the time instants $t = 0, T/2$. An SPM of period $T/2$ exists under the necessary and sufficient conditions

$$v_s(u_1^0, \ldots, u_l^0, \tau) = 0, \quad \tau = 0, T/2; \quad s = 1, \ldots, n, \tag{20}$$

where $u^0 \in M$ is the value on the SPM. Let us introduce the matrix

$$A(u^0, T/2) = ||a_{sj}|| = \left\| \frac{\partial v_s(u_1^0, \ldots, u_l^0, T/2)}{\partial u_j^0} \right\|.$$

Definition 4. The case $\det A(u^0, T/2) \neq 0$ is said to be nondegenerate for an SPM, and the latter is called a nondegenerate SPM.

An SPM is a periodic solution. By Definition 4, the nondegenerate SPMs form a family on which the period varies monotonically: Definition 1 is valid for it. Inequality (20) also holds on the period. Conditions (3) written for an SPM are satisfied identically in the $(l - n)$ values u_j^0. Hence, the matrix A_f in (5) contains $(l - n)$ simple zero eigenvalues. Therefore, the nondegeneracy condition $\operatorname{rank} A_f = \mathrm{Ra} = l + n - 1$ introduced for the general-form system (2) is true for a reversible mechanical system only if $l = n$. Accordingly, Theorem 1 applies to an SPM only in the case $l = n$.

On the other hand, without the nondegeneracy condition $\mathrm{Ra} = l + n - 1$ for a reversible mechanical system, Definition 3 remains valid for an SPM in the general situation. Therefore, Theorem 1 on the global family is true in the following formulation.

Theorem 4. *A nondegenerate SPM of a reversible mechanical system always extends to a global family of nondegenerate SPMs described by a reduced reversible mechanical system with the vector $u \in \mathrm{R}^{l-n}$ and scalar v as the variables.*

The most complete proof of Theorem 4 was given in [10]. An adaptive oscillation stabilization scheme for a reduced reversible mechanical system was designed in [11].

Theorem 4 settles an important case of SPMs, which is degenerate for Theorem 1.

6. CONSERVATIVE SYSTEM

For definiteness, assume that a conservative system is given by the Lagrange equations of the second kind and subjected to the action of potential forces. Then it is described by a system of second-order equations. Let q and $\dot{q}$ denote the coordinate and velocity vectors, respectively. Then the dynamic equations are invariant with respect to the change of variables $(q, \dot{q}, t) \to (q, \dot{q}, -t)$. Therefore, a conservative system belongs to the class of reversible mechanical systems with the coinciding dimensions of the vectors q and $\dot{q}$. The equilibria of the system belong to its fixed set. The nondegenerate equilibria are divided into centers and saddles. According to Lyapunov's center theorem [12], a local family of nonlinear periodic motions (a Lyapunov family) adjoins to a center. For such a family, the zero Jordan block has dimension $k = 2$. Due to the energy integral present in the system, the period on the family will be a monotonic function of the constant energy h. Therefore, the Lyapunov family consists of nondegenerate periodic motions. By Theorem 1, it extends to a global family of nondegenerate periodic solutions, with the reduced system containing two first-order equations. The same conclusion follows from Theorem 4, but the latter specifies that the reduced system is a reversible mechanical system. Due to the conservativeness of the original system, we obtain a reduced conservative (RCC) system with one degree of freedom; see [13, Lemma A.1].

Thus, the Lyapunov families of a conservative system always extend to global families of symmetric periodic motions: the global Lyapunov center theorem is valid, first derived in [3].

A conservative system can be described by equations not belonging to the class of reversible mechanical systems. In this case, applying Theorem 1 to the system also yields an RCC system: the energy integral is preserved.

For an RCC system, a controlled system stabilizing almost all oscillations was constructed in [7, Theorem 1]. As it turns out, all oscillations of the family are stabilized in the reduced conservative system.

Consider a controlled RCC system of the form

$$\ddot{x} + f(x) = \varepsilon\sigma(1 - Kx^2)\dot{x} \tag{21}$$

that contains a parameter K and admits, for $\varepsilon = 0$, the energy integral

$$\frac{\dot{x}^2}{2} + \int f(x)dx = h(\text{const}).$$

(Here, σ takes value 1 or (-1).) By assumption, for $\varepsilon = 0$ system (21) admits a family of nondegenerate periodic motions $x = \varphi(h, t)$.

An attracting cycle close to an oscillation with a system parameter $h = h^*$ exists under the necessary and sufficient conditions

$$I(h) \equiv \int\limits_0^{T(h^*)} [1 - K(h^*)\varphi^2(h, t)]\dot{\varphi}(h, t)dt = 0$$

(the simple root of the amplitude equation). The function $K(h)$ is calculated using the formula

$$K(h) = \frac{\int\limits_0^{T(h)} \dot{\varphi}^2(h, t)dt}{\int\limits_0^{T(h)} \varphi^2(h, t)\dot{\varphi}^2(h, t)dt}, \tag{22}$$

and the inequality $dK(h^*)/dh \neq 0$ ensures the simple root of the amplitude equation.

Theorem 5. *System* (21) *with the parameter K stabilizes any oscillation close to that of a conservative system with one degree of freedom.*

Proof. With the new time variable $\tau = t/T(h)$, the oscillation period becomes independent of h and equal to 1, whereas formula (22) is written as

$$K(h) = \frac{\int\limits_0^1 z^2(\tau)d\tau}{\int\limits_0^1 \varphi^2(h, T(h)\tau)z^2(\tau)d\tau}.$$

On the family of oscillations, the function $\varphi^2(h,t)$ with fixed t monotonically depends on h. This is true for any family, nondegenerate or degenerate. Hence, the function $K(h)$ is monotonic on the family of oscillations of a conservative system.

Thus, the sufficient condition in [7, Theorem 1] always holds for any family (nondegenerate and degenerate), and the proof of Theorem 5 is complete.

Remark 5. For a degenerate family, the function $K(h)$ is calculated in explicit form; for details, see [13].

Remark 6. According to Theorem 5, the adaptive scheme provides a complete solution of the stabilization problem for any oscillation from the family of an RCC system.

Remark 7. The result remains valid for a conservative system with an arbitrary number of degrees of freedom.

7. APPLICATIONS

1. The bounded planar three-body problem is described by the equations [14]

$$\begin{gathered}\ddot{x} - 2\dot{y} = \frac{\partial\Omega}{\partial x}, \quad \ddot{y} + 2\dot{x} = \frac{\partial\Omega}{\partial y}, \\ \Omega = \frac{1}{2}(x^2+y^2) + \frac{1-\mu}{r_0} + \frac{\mu}{r_1}, \quad \Omega(x,y) = \Omega(x,-y), \\ r_0^2 = (x+\mu)^2 + y^2, \quad r_1^2 = (x-1+\mu)^2 + y^2.\end{gathered} \tag{23}$$

They contain a single dimensionless mass parameter μ.

System (23) admits an energy integral. Due to its invariance with respect to the transformation

$$\{x, y, \dot{x}, \dot{y}, t\} \to \{x, -y, -\dot{x}, \dot{y}, -t\},$$

system (23) also belongs to the class of reversible mechanical systems.

The relative equilibria (libration points) are found from the equations

$$\frac{\partial\Omega}{\partial x} = 0, \quad \frac{\partial\Omega}{\partial y} = 0.$$

The problem admits five libration points L_i, $i = 1, \ldots, 5$. In addition, L_1, L_2, and L_3 lie on the abscissa axis x, whereas the points L_4 and L_5 are located symmetrically with respect to the axis x and form equilateral triangles with the main bodies on the axis x.

The points L_1, L_2, and L_3 belong to the equilibria of a reversible mechanical system. They are adjoined by a symmetric Lyapunov family for which the matrix A_f (Section 2) has a zero second-order Jordan block. Therefore, Theorem 4 can be applied here. The global family is described by an conservative system with one degree of freedom and symmetric orbits. By Theorem 5, the orbital stabilization problem of any orbit is solved by the adaptive control scheme.

Theorem 1 can be applied to the points L_4 and L_5. System (23) is conservative, so the global family is described by an RCC system. Any orbit of the global family is stabilized using Theorem 5.

The bounded three-body problem is basic in the theory of orbits [15]. In the theory of controlled orbital motion, the problem of orbital stabilization is solved. The corresponding results will be considered in detail in a separate paper.

2. Stabilization of satellite oscillations. Under gravitational forces, the motion of a satellite in an orbital plane is described by V.V. Beletskii's equation [5]. Consider a particular case of a circular orbit. The corresponding Beletskii equation takes the form

$$\ddot{\alpha} + \mu \sin\alpha \cos\alpha = 0, \quad \dot{\alpha} = \frac{d\alpha}{dv}, \tag{24}$$

where μ is the inertial parameter ($||\mu| \leqslant 3$); α is the angle between the radius vector of the center of mass and the main central axis of inertia of the satellite in the orbital plane; v is the true anomaly chosen as the independent variable. As a result, we obtain the mathematical pendulum equation

$$\ddot{y} + \mu \sin y = 0, \quad \mu > 0, \quad y = 2\alpha,$$

or

$$\ddot{y} + |\mu| \sin y = 0, \quad \mu < 0, \quad y = 2\alpha + \pi.$$

The satellite oscillations form a family from an initial deviation in the angle y, and the period $T(h)$ increases on this family.

A mechatronic oscillation stabilization scheme with a van der Pol oscillator was proposed in [13]. The mechatronic scheme is adaptive in the sense of this paper.

According to Theorem 5, any oscillation is locally stabilized using the adaptive scheme

$$\ddot{x} + |\mu| \sin x = \varepsilon\sigma(1 - Kx^2)\dot{x},$$

where $x = 2\alpha$ and $\mu > 0$ or $x = 2\alpha + \pi$ and $\mu < 0$, and $\sigma = 1$. Here, the van der Pol oscillator is not applied.

Note that for the problem of rotational motion of a satellite, an equilibrium stabilization scheme was proposed, e.g., in [16].

8. CONCLUSIONS

In the nondegenerate case, the periodic solution of an autonomous system of general form can be a cycle or belong to a family. For a cycle, the Jacobi matrix has one zero eigenvalue, whereas a family corresponds to a zero kth-order Jordan block. For $k = 2$, the case usually considered, a global family of nondegenerate periodic solutions was constructed in [3]; this family is described by a reduced second-order system. The results are valid for the general case of dimension $1 < k \leqslant n$, where n denotes the dimension of the autonomous system. An adaptive oscillation stabilization scheme is designed for the reduced system of order k. An autonomous control with the parameter $K(h)$ depending on the global family parameter h is applied. It represents a generalization of the universal control [7] to a system of arbitrary order; the value of h separates an oscillation in the global family. Stabilization is achieved by implementing an attracting cycle.

For an nth-order differential equation in the variable x, the reduced system coincides with the original one. The periodic solution is stabilized by a control law that represents nonlinear dissipation, being linear in velocity, acting in the neighborhood of the cycle, containing a parameter, and ensuring attraction to the cycle in the space $(x, x', \ldots, x^{(n-1)})$.

Similar results are known for symmetric periodic motions of reversible mechanical systems [10]. In the spatiotemporal symmetry case, the Jacobi matrix admits zero eigenvalues: simple ones and those of a single 2nd-order Jordan block. This block corresponds to a conservative system. The stabilization problem of family oscillations finds an exhaustive solution for the conservative system with one degree of freedom: the stabilization conditions of the selected oscillation are automatically satisfied in the controlled system.

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